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The generation of internal waves by vibrating elliptic cylinders. Part 1. Inviscid solution

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We consider the internal gravity waves that are produced in an inviscid Boussinesq fluid, whose Brunt–Väisälä frequency N is constant, by the small rectilinear vibrations of a horizontal elliptic cylinder whose major axis is inclined at an arbitrary angle to the horizontal. When the angular frequency ω is greater than N, no waves are produced and the governing elliptic equation is solved using conformal transformations. Analytic continuation in ω to values less than N, when waves are produced, is then used to determine the solution. It exhibits the surprising feature that, apart from certain phase differences, the form of the velocity distributions in each of the beams of waves that occur is the same for all values of the thickness ratio of the ellipse, the inclination of its major axis to the horizontal and the plane in which the vibrations are occurring. The Fourier decomposition of the velocity distribution is found and is used in a sequel, Part 2, to investigate the effects of viscous dissipation.

In an important paper Makarov *et al.* (1990) have given an approximate solution for a vibrating circular cylinder in a viscous fluid. We show that the limit of this solution as the viscosity tends to zero is not the exact inviscid solution discussed herein. Further comparison of their work and ours will be made in Part 2.

1. Introduction

There appear to be few solutions to the problem of calculating the internal gravity waves that are generated by a vibrating body in a stably stratified fluid, in spite of its relevance to both oceanography and atmospheric physics. When the fluid is assumed to be inviscid and the Boussinesq approximation is made, solutions have been obtained for the sphere (Hendershott 1969; Appleby & Crighton 1987), the flat plate (Hurley 1969), and the circular cylinder (Appleby & Crighton 1986) who also considered non-Boussinesq effects. Also, Voisin (1991) gives solutions to a variety of three-dimensional problems and also a comprehensive bibliography.

Mention should also be made of the important work of Robinson (1969, 1970). He considered the inviscid problem of an internal wave in a horizontal channel incident on a vertical barrier; he determined the solution that satisfies the radiation condition and introduced the 'vortex function' of which good use will be made in Part 2 (Hurley & Keady 1997). Robinson's success was dependent on crucial advice from the late Professor J. J. Mahony referred to on p. 5 of Robinson (1970) and described in greater detail in Fowkes & Silberstein (1995, p. 277).

For a viscous fluid the pioneering work of Mowbray & Rarity (1967) was followed by that of Thomas & Stevenson (1972) who described a similarity solution that holds at large distances from the body. Lighthill (1978) gives a comprehensive account of the various types of waves that occur in fluids and pays particular attention to internal gravity ones. Numerous theoretical and experimental contributions have also been made by Russian workers. Particular mention should be made of the paper by Makarov, Neklyudov & Chasheckin (1990), which reviews this work and in particular describes theoretical and experimental work on a vibrating circular cylinder.

More recent papers on the linear theory are referenced in Kistovich & Chasheckin (1995), and in Kistovich, Neklyudov & Chasheckin (1990) nonlinear effects are studied.

The present investigation consists of two parts. In Part 1, the present paper, we investigate the waves that are produced in an inviscid Boussinesq fluid whose Brunt–Väisälä frequency N is constant. The waves are produced by the small rectilinear vibrations at angular frequency ω of a horizontal elliptic cylinder whose major axis is inclined at an arbitrary angle to the horizontal. We follow the approach described in Hurley (1972) by first considering the case $\omega > N$ and then use analytic continuation in ω to find the solution when $\omega < N$.

The problem for $\omega > N$ is considered in §2. A simple (non-conformal) affine transformation reduces the governing differential equation for the stream function to Laplace's equation and the given ellipse is transformed into a strained one. The solution is then obtained by conformally mapping the strained ellipse onto a circle. In §3 the analytic continuation in ω to values less than N is carried out to obtain the solution in that case. In the special cases of a flat plate and a circular cylinder the solutions agree with those previously published. In the final section, §4, our results are compared with those of other investigators.

The solution obtained herein has two features that limit its applicability to a real fluid of small viscosity. First, the fluid velocities have inverse-square-root singularities at all points of the characteristics that touch the ellipse so that the kinetic energy per unit length of characteristic is infinite. Also the velocities do not decay with increasing distance from the ellipse.

Part 2 (Hurley & Keady 1997) describes an attempt to overcome these deficiencies. The Fourier decomposition of the stream function found in Part 1 is modified by including in the integrands factors to account for viscous dissipation. These factors tend to unity as the viscosity tends to zero so that the exact inviscid solution is obtained in this limit.

2. Solution for $\omega > N$

Consider the two-dimensional motions produced in an inviscid stably stratified fluid of constant Brunt–Väisälä frequency N by the small rectilinear vibrations of a rigid elliptic cylinder. The elliptic cylinder has semi-axes of lengths a and b, the a semi-axis being inclined at an angle θ to the horizontal. We introduce Cartesian axes Oxy, Ox being horizontal and Oy vertically up. We suppose that the velocity of each point of the surface of the cylinder is $(U, V)\exp(-\mathrm{i}\omega t)$ where t is the time and the angular frequency $w > N$. The notation is shown in figure 1.

The equation of the ellipse, relative to the principal axes Ox_0y_0, is

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2} = 1, \tag{2.1}$$

where

$$x_0 = x\cos\theta+y\sin\theta, \quad y_0 = -x\sin\theta+y\cos\theta, \tag{2.2}$$

so that, relative to the Oxy axes, its equation is

$$x^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+y^2\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\right)+2xy\sin\theta\cos\theta\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = 1. \tag{2.3}$$

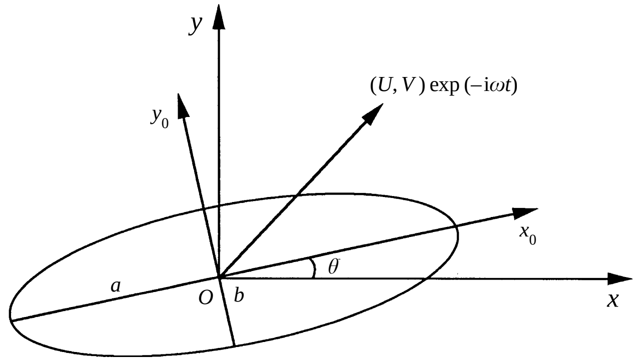

FIGURE 1. Notation.

The resulting fluid motions may be described in terms of the stream function $\psi(x, y)\exp(-\mathrm{i}\omega t)$ such that

$$u = -\frac{\partial \psi}{\partial y}\exp(-\mathrm{i}\omega t), \quad v = \frac{\partial \psi}{\partial x}\exp(-\mathrm{i}\omega t), \tag{2.4}$$

and

$$\alpha^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{2.5}$$

where

$$\alpha^2 = 1 - N^2/\omega^2. \tag{2.6}$$

In addition u and v must decay (as much as possible) at large distances from the ellipse and ψ must satisfy the boundary condition

$$\psi = Vx - Uy \tag{2.7}$$

on the ellipse given by (2.3). Under the transformation

$$x_1 = x/\alpha, \quad y_1 = y, \tag{2.8}$$

equation (2.5) becomes Laplace's equation,

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial y_1^2} = 0 \tag{2.9}$$

and the ellipse given by (2.3) transforms to another ellipse whose equation in the $z_1 = x_1 + \mathrm{i}y_1$ plane is

$$a_{11}x_1^2 + a_{22}y_1^2 + 2a_{12}x_1y_1 = 1, \tag{2.10}$$

where

$$a_{11} = \alpha^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right), \quad a_{22} = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}, \quad a_{12} = \alpha\sin\theta\cos\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right). \tag{2.11}$$

Also the boundary condition (2.7) becomes

$$\psi = \alpha V x_1 - U y_1 \tag{2.12}$$

on the ellipse given by (2.10).

The lengths a_1 and b_1 of the semi-axes of the ellipse (2.10) and the inclination θ_1 of the a_1 semi-axis to Ox_1 may be found by the method of Jeffreys & Jeffreys (1956). We find that, in terms of the quantities defined by (2.11),

$$\left.\begin{aligned} a_1^2 &= \frac{a_{11}+a_{22}+\Delta^{1/2}}{2(a_{11}a_{22}-a_{12}^2)}, \quad b_1^2 = \frac{a_{11}+a_{22}-\Delta^{1/2}}{2(a_{11}a_{22}-a_{12}^2)}, \\ \theta_1 &= \arctan\left(\frac{a_{22}-a_{11}-\Delta^{1/2}}{2a_{12}}\right), \end{aligned}\right\} \tag{2.13}$$

where

$$\Delta = (a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}^2). \tag{2.14}$$

Under the transformation

$$z_2 = z_1 \exp(-\mathrm{i}\theta_1), \tag{2.15}$$

the ellipse in the z_1-plane is rotated through an angle $-\theta_1$ to make its axes coincide with Ox_2 and Oy_2. We now apply the Joukowski transformation

$$z_2 = z_3 + \frac{a_1^2-b_1^2}{4z_3} \tag{2.16}$$

which maps the region outside the ellipse in the z_2-plane onto the region outside the circle in the z_3-plane of radius

$$a_3 = \tfrac{1}{2}(a_1+b_1). \tag{2.17}$$

The inverse of the transformation (2.16) is

$$z_3 = \tfrac{1}{2}\{z_2 + [z_2^2 - (a_1^2-b_1^2)]^{1/2}\}. \tag{2.18}$$

where $[z_2^2-(a_1^2-b_1^2)]^{1/2}$ behaves as z_2 as $|z_2| \to \infty$. Hence the most general form for ψ that represents motions in which the fluid velocities decay at large distances is

$$\psi = k_0 \log z_3 + \sum_{n=1}^{\infty} k_n z_3^{-n} + \bar{k}_0 \log \bar{z}_3 + \sum_{n=1}^{\infty} \bar{k}_n \bar{z}_3^{-n}, \tag{2.19}$$

where the overbar denotes the complex conjugate.

It remains for us to determine the coefficients k_n so that ψ given by (2.19) satisfies the boundary condition (2.12).

On the circle in the z_3-plane, we take

$$z_3 = \tfrac{1}{2}(a_1+b_1)\exp(\mathrm{i}\ \ ) \tag{2.20}$$

and (2.15) and (2.16) give

$$x_1 = a_1 \cos\ \ \cos\theta_1 - b_1 \sin\ \ \sin\theta_1, \quad y_1 = a_1\cos\ \ \sin\theta_1 + b_1 \sin\ \ \cos\theta_1. \tag{2.21}$$

Substitution into the boundary condition (2.12) and equating coefficients of sin and cos shows that it is satisfied provided

$$k_n = 0, \quad n \neq 1,$$

and

$$k_1 = \tfrac{1}{4}(a_1+b_1)\{\alpha V a_1 \cos\theta_1 - U a_1 \sin\theta_1 - \mathrm{i}(\alpha V b_1 \sin\theta_1 + U b_1 \cos\theta_1)\}. \tag{2.22}$$

Hence the desired solution is

$$\psi = \frac{k_1}{z_3} + \frac{\bar{k}_1}{\bar{z}_3}, \tag{2.23}$$

where z_3 is given by (2.18) and k_1 by (2.22).

3. Solution for $\omega < N$

When $\omega < N$ the governing equation for ψ is

$$\frac{\partial^2 \psi}{\partial y^2} - \eta^2 \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{3.1}$$

where

$$\eta^2 = N^2/\omega^2 - 1. \tag{3.2}$$

We introduce the coordinates

$$\sigma_+ = x \sin\mu - y\cos\mu, \quad \sigma_- = x\sin\mu + y\cos\mu, \tag{3.3}$$

where

$$\eta = \cot\mu. \tag{3.4}$$

In terms of them, (3.1) becomes

$$\frac{\partial^2 \psi}{\partial\sigma_+ \partial\sigma_-} = 0 \tag{3.5}$$

whose general solution is

$$\psi = \psi_+(\sigma_+) + \psi_-(\sigma_-). \tag{3.6}$$

Our approach is to use analytic continuation in ω to find the functions $\psi_+(\sigma_+)$ and $\psi_-(\sigma_-)$ from the solution for $\omega > N$ given by (2.22) and (2.23). This solution for ψ is of the form

$$\psi = \psi(a, b, \theta, U, V, \alpha) \tag{3.7}$$

and the only dependence on ω is through α which is defined by (2.6). Under the analytic continuation it becomes $\mathrm{i}\eta$, where η is defined by (3.2), so that the solution for $\omega < N$ is

$$\psi_c = \psi(a, b, \theta, U, V, \mathrm{i}\eta), \tag{3.8}$$

where we have introduced the notation that subscript c denotes the outcome of the analytic continuation.

First, we determine $(z_3)_c$ where z_3 is defined by (2.18). Referring to (2.13) we have

$$(a_1^2)_c = \frac{(a_{11}+a_{22})_c + (\Delta^{1/2})_c}{2(a_{11}a_{22}-a_{12}^2)_c}, \quad (b_1^2)_c = \frac{(a_{11}+a_{22})_c - (\Delta^{1/2})_c}{2(a_{11}a_{22}-a_{12}^2)_c}. \tag{3.9}$$

Then (2.11) readily show that

$$(a_{11}+a_{22})_c = \frac{a^2\sin(\mu-\theta)\sin(\mu+\theta) - b^2\cos(\mu-\theta)\cos(\mu+\theta)}{a^2b^2\sin^2\mu}, \tag{3.10}$$

$$(a_{11}a_{22}-a_{12}^2)_c = -\frac{\cos^2\mu}{a^2b^2\sin^2\mu}, \tag{3.11}$$

and

$$\Delta_c = \frac{1}{a^4b^4\sin^4\mu}\Big\{a^4\sin^2(\mu-\theta)\sin^2(\mu+\theta) + b^4\cos^2(\mu-\theta)\cos^2(\mu+\theta) + \tfrac{1}{2}a^2b^2\sin^2 2\mu + \sin^2 2\theta)\Big\}. \tag{3.12}$$

We now simplify (3.12) by using the important identity

$$a^4\sin^2(\mu-\theta)\sin^2(\mu+\theta) + b^4\cos^2(\mu-\theta)\cos^2(\mu+\theta) + \tfrac{1}{2}a^2b^2(\sin^2 2\mu + \sin^2 2\theta) = c_+^2 c_-^2, \tag{3.13}$$

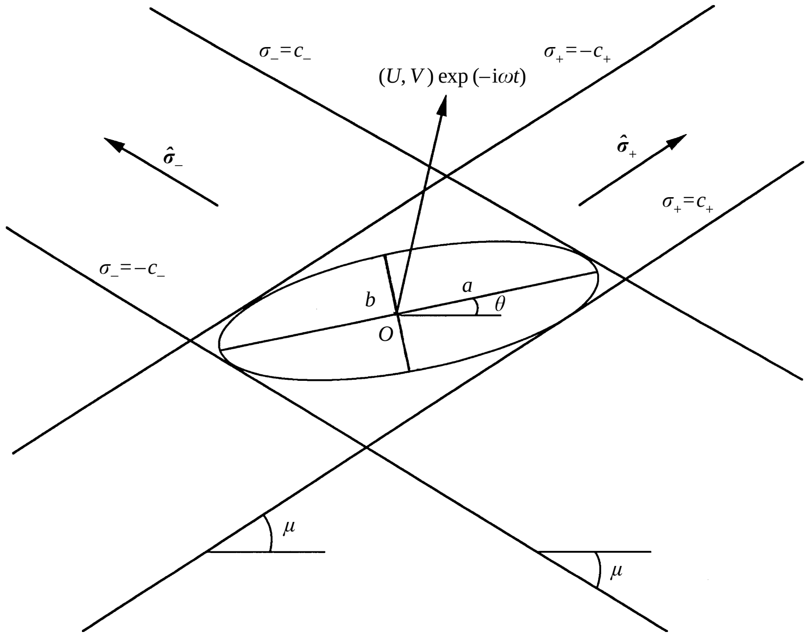

FIGURE 2. Notation.

where

$$\left.\begin{aligned} c_+^2 &= a^2 \sin^2(\mu-\theta) + b^2\cos^2(\mu-\theta), \\ c_-^2 &= a^2 \sin^2(\mu+\theta) + b^2\cos^2(\mu+\theta). \end{aligned}\right\} \tag{3.14}$$

Equations (3.12) and (3.13) now give

$$(\Delta^{1/2})_c = \frac{c_+ c_-}{a^2 b^2 \sin^2\mu}. \tag{3.15}$$

It can be shown using analytic geometry (see, for example, Sommerville 1937) that the straight lines $\sigma_+ = \pm c_+$ and $\sigma_- = \pm c_-$, with c_+ and c_- given by (3.14), are tangential to the ellipse as shown in figure 2.

Equations (3.9), (3.11) and (3.15) now give

$$(a_1^2 - b_1^2)_c = -c_+ c_-/\cos^2\mu \tag{3.16}$$

and

$$[(a_1^2 - b_1^2)^{1/2}]_c = \pm \mathrm{i} c_+^{1/2} c_-^{1/2}/\cos\mu. \tag{3.17}$$

For the special case $\theta = 0$, $b = 0$, equation (2.8) gives $a_1 = a/\alpha$, so that $(a_1)_c = -\mathrm{i}a\sin\mu/\cos\mu$ and we conclude that the minus sign in (3.17) is to be taken so that

$$[(a_1^2 - b_1^2)^{1/2}]_c = -\mathrm{i} c_+^{1/2} c_-^{1/2}/\cos\mu. \tag{3.18}$$

To determine $(\theta_1)_c$ where θ_1 is defined by (2.13) we use the result

$$\arctan z = \tfrac{1}{2}\mathrm{i}\log\frac{1-\mathrm{i}z}{1+\mathrm{i}z} \tag{3.19}$$

and find that

$$(\exp \mathrm{i}\theta_1)_c = (c_+/c_-)^{1/2}, \quad (\exp -\mathrm{i}\theta_1)_c = (c_-/c_+)^{1/2} \tag{3.20}$$

so that

$$(\cos\theta_1)_c = \tfrac{1}{2}\{(c_+/c_-)^{1/2} + (c_-/c_+)^{1/2}\}$$

and

$$(\sin\theta_1)_c = \tfrac{1}{2}\mathrm{i}\{(c_+/c_-)^{1/2} - (c_-/c_+)^{1/2}\}. \tag{3.21}$$

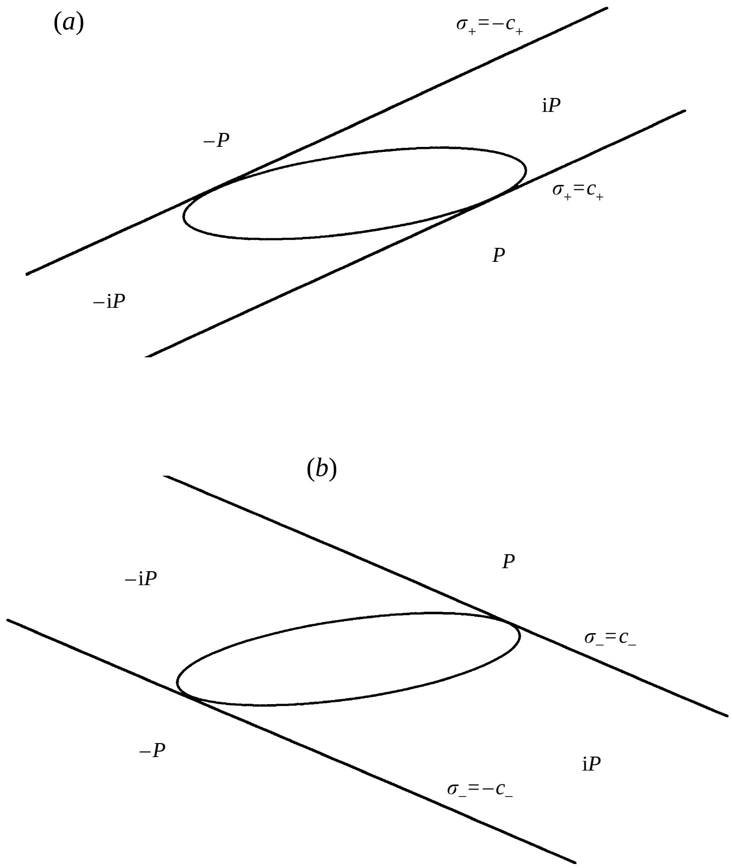

FIGURE 3. (*a*) Values of $(\sigma_+^2/c_+^2-1)^{1/2}$ outside the ellipse where P denotes the positive real number $|\sigma_+^2/c_+^2-1|^{1/2}$. (*b*) Values of $(\sigma_-^2/c_-^2-1)^{1/2}$ outside the ellipse where P denotes the positive real number $|\sigma_-^2/c_-^2-1|^{1/2}$.

Using results above, we now find that

$$(z_3)_c = \frac{-\mathrm{i}c_+^{1/2}\,c_-^{1/2}}{2\cos\mu}\left\{\frac{\sigma_+}{c_+}+\left[\frac{\sigma_+^2}{c_+^2}-1\right]^{1/2}\right\} \tag{3.22}$$

where $[\sigma_+^2/c_+^2-1]^{1/2}$ behaves as σ_+/c_+ as $\sigma_+\to+\infty$. To determine the values of $[\sigma_+^2/c_+^2-1]^{1/2}$ in the various regions of the Oxy plane we use the procedure described in Hurley (1972). If ω in (3.2) is replaced by $\omega+\mathrm{i}\epsilon$ we find that both the real and imaginary parts of ψ satisfy an elliptic equation. Its solution $\psi(x,y;\omega+\mathrm{i}\epsilon)$ will therefore be analytic in x and y and analytic continuation in x and y may be used to determine the values of the square root in the various regions of the Oxy-plane. Taking the limit $\epsilon\to 0$ gives the values shown in figure 3.

To calculate $(\bar{z}_3)_c$ we note that (2.15) and (3.20) give

$$(\bar{z}_2)_c = -\frac{\mathrm{i}\sigma_-}{\cos\mu}\left(\frac{c_+}{c_-}\right)^{1/2}. \tag{3.23}$$

Equations (2.19) and (3.19) then give

$$(\bar{z}_3)_c = \frac{-\mathrm{i}c_+^{1/2}\,c_-^{1/2}}{2\cos\mu}\left\{\frac{\sigma_-}{c_-}+\left[\frac{\sigma_-^2}{c_-^2}-1\right]^{1/2}\right\}, \tag{3.24}$$

and the values of $[\sigma_-^2/c_-^2-1]^{1/2}$ are included in figure 3.

The calculations of $(k_1)_c$ and $(\bar{k}_1)_c$ where k_1 is given by (2.22) are straight-forward, and, combining the results with those of (3.22) and (3.24), we find that

$$\psi = \frac{c_+ \alpha_+}{\sigma_+/c_+ + (\sigma_+^2/c_+^2 - 1)^{1/2}} + \frac{c_- \alpha_-}{\sigma_-/c_- + (\sigma_-^2/c_-^2 - 1)^{1/2}}, \quad \omega < N, \tag{3.25}$$

where

$$\alpha_+ = \frac{1}{2c_+^2}\left\{a^2(V\cos\theta - U\sin\theta)\sin(\mu-\theta) + b^2(V\sin\theta + U\cos\theta)\cos(\mu-\theta) \right. \\ \left. + \mathrm{i}ab(V\cos\mu - U\sin\mu)\right\} \tag{3.26}$$

and

$$\alpha_- = \frac{1}{2c_-^2}\left\{a^2(V\cos\theta - U\sin\theta)\sin(\mu+\theta) - b^2(V\sin\theta + U\cos\theta)\cos(\mu+\theta) \right. \\ \left. + \mathrm{i}ab(V\cos\mu + U\sin\mu)\right\}. \tag{3.27}$$

As an overall check on the algebra it is verified in the Appendix that the solution given by equations (3.25)–(3.27) does satisfy the boundary condition (2.7).

The fluid velocities, $\tilde{\boldsymbol{V}}$, given by (3.25), can be expressed in the form

$$\tilde{\boldsymbol{V}} = \left\{\frac{\partial\psi}{\partial\sigma_+}\hat{\boldsymbol{\sigma}}_+ + \frac{\partial\psi}{\partial\sigma_-}\hat{\boldsymbol{\sigma}}_-\right\}\exp(-\mathrm{i}\omega t) \tag{3.28}$$

where $\hat{\boldsymbol{\sigma}}_+$ and $\hat{\boldsymbol{\sigma}}_-$ are unit vectors in the directions shown in figure 2. Equation (3.25) gives

$$\frac{\partial\psi}{\partial\sigma_+} = \alpha_+\left\{1 - \frac{\sigma_+/c_+}{(\sigma_+^2/c_+^2 - 1)^{1/2}}\right\} \quad \text{and} \quad \frac{\partial\psi}{\partial\sigma_-} = \alpha_-\left\{1 - \frac{\sigma_-/c_-}{(\sigma_-^2/c_-^2 - 1)^{1/2}}\right\}, \tag{3.29}$$

the values of the square roots being given in figure 3. Hence, for example, the values of $\partial\psi/\partial\sigma_+$ for the beam of waves that lies in the first quadrant are

$$\frac{\partial\psi}{\partial\sigma_+} = \begin{cases} \alpha_+\left\{1 - \dfrac{\sigma_+/c_+}{(\sigma_+^2/c_+^2 - 1)^{1/2}}\right\}, & \dfrac{\sigma_+}{c_+} > 1, \\[2ex] \alpha_+\left\{1 + \mathrm{i}\dfrac{\sigma_+/c_+}{(1 - \sigma_+^2/c_+^2)^{1/2}}\right\}, & -1 < \dfrac{\sigma_+}{c_+} < 1, \\[2ex] \alpha_+\left\{1 + \dfrac{\sigma_+/c_+}{(\sigma_+^2/c_+^2 - 1)^{1/2}}\right\}, & \dfrac{\sigma_+}{c_+} < -1. \end{cases} \tag{3.30}$$

Values of the velocity distribution function $(1/\alpha_+)(\partial\psi/\partial\sigma_+)$, which is the same for all values of a, b, θ, U and V, are shown in figure 4.

Using results given in Erdélyi *et al.* (1954) we find that, for waves in the first quadrant,

$$\psi_+ = -\mathrm{i}\alpha_+ c_+ \int_0^\infty \frac{J_1(K)}{K}\exp\left(\mathrm{i}K\frac{\sigma_+}{c_+}\right)\mathrm{d}K, \tag{3.31}$$

and

$$\frac{\partial\psi}{\partial\sigma_+} = \alpha_+ \int_0^\infty J_1(K)\exp\left(\mathrm{i}K\frac{\sigma_+}{c_+}\right)\mathrm{d}K, \tag{3.32}$$

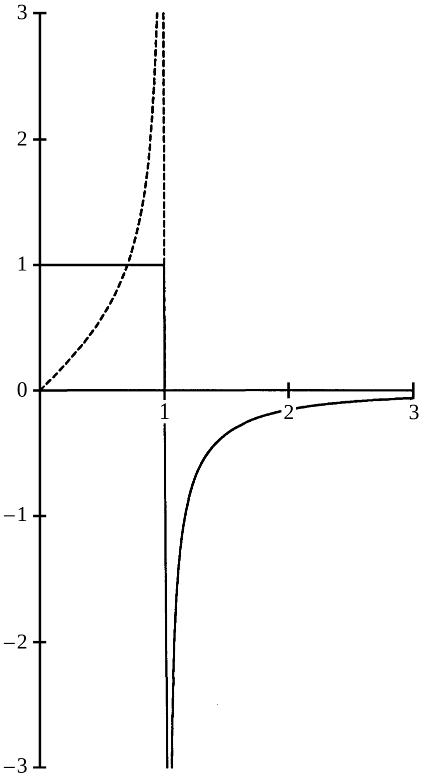

FIGURE 4. The velocity distribution function $(1/\alpha_+)(\partial\psi/\partial\sigma_+)$ (see (3.30)): ——, real part; – – – – –, imaginary part.

where $J_1(K)$ is a Bessel function of the first kind. This result confirms that the radiation condition is satisfied.

For waves in the third quadrant, figure 1 of Hurley (1969) shows that only negative values of K satisfy the radiation condition, so that

$$\psi_+ = \mp \mathrm{i}\alpha_+ c_+ \int_0^\infty \frac{J_1(K)}{K} \exp\left(\pm \mathrm{i}K\frac{\sigma_+}{c_+}\right) \mathrm{d}K, \quad \text{for waves in the } \frac{\text{1st}}{\text{3rd}} \text{ quadrant resp.} \tag{3.33}$$

The corresponding results for ψ_- are

$$\psi_- = \pm \mathrm{i}\alpha_- c_- \int_0^\infty \frac{J_1(K)}{K} \exp\left(\mp \mathrm{i}K\frac{\sigma_-}{c_-}\right) \mathrm{d}K, \quad \text{for waves in the } \frac{\text{2nd}}{\text{4th}} \text{ quadrant resp.} \tag{3.34}$$

Useful alternative expressions for ψ_+ are, using figures 3 and 5,

$$\psi_+ = \begin{cases} c_+\alpha_+\left[\dfrac{\sigma_+}{c_+} - \left(\dfrac{\sigma_+^2}{c_+^2} - 1\right)^{1/2}\right], s_+ = 0+ \quad \text{and} \quad 0-, \sigma_+ > c_+, & (3.35) \\ c_+\alpha_+\left[\dfrac{\sigma_+}{c_+} + \left(\dfrac{\sigma_+^2}{c_+^2} - 1\right)^{1/2}\right], s_+ = 0+ \quad \text{and} \quad 0-, \sigma_+ < -c_+, & (3.36) \\ c_+\alpha_+\left[\dfrac{\sigma_+}{c_+} \mp \mathrm{i}\left(1 - \dfrac{\sigma_+^2}{c_+^2}\right)^{1/2}\right], s_+ = 0\pm, \quad -c_+ < \sigma_+ < c_+. & (3.37) \end{cases}$$

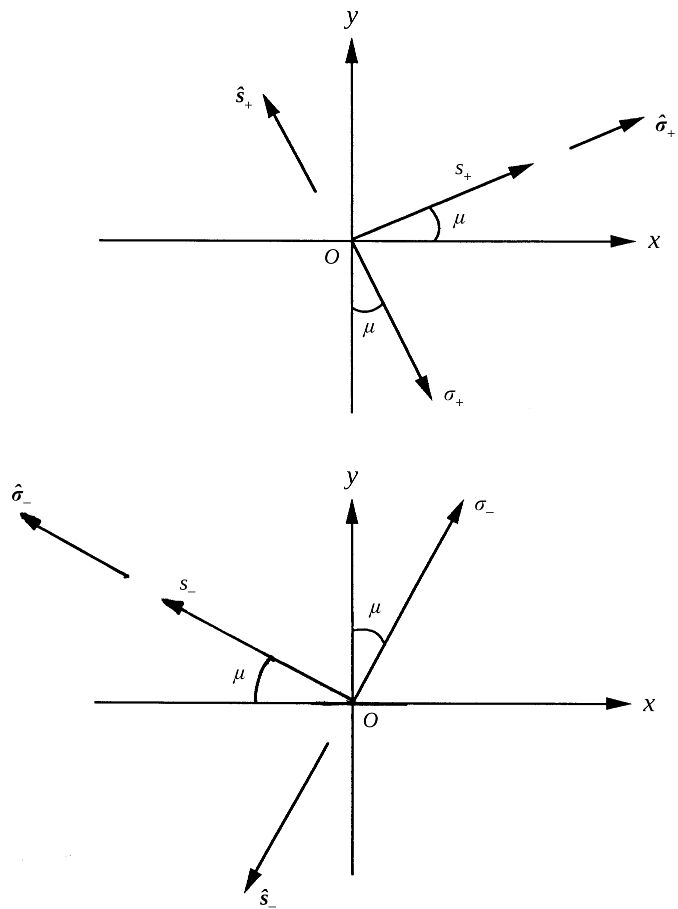

FIGURE 5. Notation.

(The square roots in equations (3.35)–(3.37) denote positive quantities.)

Using results given in Hurley (1969), it may readily be shown that the pressure, p, corresponding to the solution (3.25) is

$$p = \mathrm{i}\rho_0\,\omega\eta\left\{c_+\,\alpha_+\left[\frac{\sigma_+}{c_+}-\left(\frac{\sigma_+^2}{c_+^2}-1\right)^{1/2}\right]-c_-\,\alpha_-\left[\frac{\sigma_-}{c_-}-\left(\frac{\sigma_-^2}{c_-^2}-1\right)^{1/2}\right]\right\}. \tag{3.38}$$

It follows that the time average of the power radiated in the beam in the first quadrant, P, is

$$\begin{aligned} P &= \frac{1}{4}\int_{-c_+}^{c_+}\left(p\frac{\partial\bar{\psi}}{\partial\sigma_+}+\bar{p}\frac{\partial\psi}{\partial\sigma_+}\right)\mathrm{d}\sigma_+ \\ &= \tfrac{1}{2}\pi\eta\rho_0\,\omega c_+^2\,\alpha_+\,\bar{\alpha}_+. \end{aligned} \tag{3.39}$$

Equation (3.38) may also be used to calculate the force per unit length, $\mathscr{F}$, exerted by the fluid on the cylinder. We find that

$$\begin{aligned} \mathscr{F} = \pi\rho_0\,\omega\eta\{&\hat{\boldsymbol{s}}_+[\mathrm{i}ab\alpha_+-(a^2-b^2)\sin(\mu-\theta)\cos(\mu-\theta)\,\alpha_+]-c_+^2\,\alpha_+\,\hat{\boldsymbol{\sigma}}_+ \\ &+\hat{\boldsymbol{s}}_-[-\mathrm{i}ab\alpha_--(a^2-b^2)\sin(\mu+\theta)\cos(\mu+\theta)\,\alpha_-]-c_-^2\,\alpha_-\,\hat{\boldsymbol{\sigma}}_-\}, \end{aligned} \tag{3.40}$$

where $\hat{\boldsymbol{s}}_+$, $\hat{\boldsymbol{\sigma}}_+$, $\hat{\boldsymbol{s}}_-$ and $\hat{\boldsymbol{\sigma}}_-$ are unit vectors in the directions shown in figure 5.

When $b = 0$ the ellipse becomes a flat plate of length $2a$ and (3.25) gives

$$\psi = \frac{aU_n}{2}\left\{\frac{\sigma_+}{\alpha \sin(\mu-\theta)} - \left(\frac{\sigma_+^2}{a^2\sin^2(\mu-\theta)} - 1\right)^{1/2} + \frac{\sigma_-}{a\sin(\mu+\theta)} - \left(\frac{\sigma_-^2}{a^2\sin^2(\mu+\theta)} - 1\right)^{1/2}\right\}, \quad (3.41)$$

where U_n is the normal velocity of the plate. This result agrees with that of Hurley (1969).

When $a = b$ the ellipse becomes a circle and we find that

$$\psi = \frac{a}{2}\left\{V\sin\mu + U\cos\mu + \mathrm{i}(V\cos\mu - U\sin\mu)\right\}\left\{\frac{\sigma_+}{a} - \left(\frac{\sigma_+^2}{a^2} - 1\right)^{1/2}\right\} + \frac{a}{2}\left\{V\sin\mu - U\cos\mu + \mathrm{i}(V\cos\mu + U\sin\mu)\right\}\left\{\frac{\sigma_-}{a} - \left(\frac{\sigma_-^2}{a^2} - 1\right)^{1/2}\right\}, \quad (3.42)$$

which agrees with results given in Appleby & Crighton (1986). Also, in this case (3.40) gives

$$\mathscr{F} = (F_x, F_y) = (-\pi\rho_0\,\omega a^2\eta U, \pi\rho_0\,\omega a^2\eta V), \quad (3.43)$$

a result that we will use in Part 2.

4. Discussion

The solution we have derived for the internal waves that are generated by a vibrating elliptic cylinder in an inviscid Boussinesq fluid exhibits the surprising feature that the velocity distribution function given by (3.30) is the same for all values of the thickness ratio of the ellipse, the inclination of its major axis to the horizontal and the plane in which the vibrations are occurring.

The wave forms, $\mathrm{Re}((\partial\psi/\partial\sigma_+)\exp(-\mathrm{i}\omega t))$, at successive instants of time are given in figure 6 for a circular cylinder executing horizontal oscillations. (We take $a = 1$, $\mu = \pi/4$, $V = 0$, $U = 1$.) They exhibit an extreme 'bimodal' form (Makarov *et al.* 1990), in that the envelope of the curves has twin maxima.

4.1. *Comparison of our results with those of Makarov et al. (1990)*

We focus on results for the circular cylinder.

Our solution is expressed in the form of (3.6), with $\psi_+(\sigma_+)$ and $\psi_-(\sigma_-)$ therein given by (3.42). For a point $x = a\cos\phi$, $y = b\sin\phi$ on the surface of the cylinder, (3.42) gives

$$\psi_+ = \tfrac{1}{2}a(V\cos\phi - U\sin\phi - \mathrm{i}(U\cos\phi + V\sin\phi), \quad (4.1)$$

$$\psi_- = \tfrac{1}{2}a(V\cos\phi - U\sin\phi + \mathrm{i}(U\cos\phi + V\sin\phi). \quad (4.2)$$

We note that neither ψ_+ nor ψ_- satisfy the boundary condition (2.7) but that their sum does.

Model 4 of Makarov *et al.* (1990) concerns approximating the solution for a vibrating circular cylinder in a viscous fluid by representing it as a distribution of dipoles on its surface. To compare it with our inviscid one we take the limit of their solution as the kinematic viscosity ν tends to zero. Their expression for the vertical displacement of the fluid particles is then, in our notation,

$$h = -\tfrac{1}{2}\exp(-\mathrm{i}\omega t)\,a_0\sin\mu\sin(\mu-\phi_0)\int_0^\infty J_1(K)\exp\left(\mathrm{i}K\frac{\sigma_+}{a}\right)\mathrm{d}K, \quad (4.3)$$

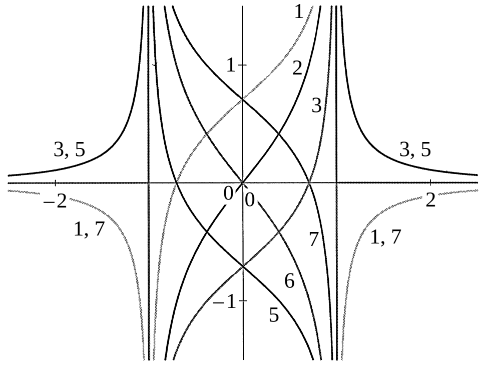

FIGURE 6. Wave forms $\mathrm{Re}((\partial\psi/\partial\sigma_+)\exp(-\mathrm{i}\omega t))$ at successive instants of time for a cylinder executing horizontal oscillations. (Calculated from (3.42) with $V = 0$.) A label j at a curve indicates a phase $j\pi/4$.

where a_0 is the amplitude of the particle displacement and ϕ_0 is its inclination to the horizontal. The particle displacement h_{σ_+} in the direction of $\hat{\boldsymbol{\sigma}}_+$ (defined in figure 5) is $h/\sin\mu$ and differentiation with respect to t gives that the particle velocity in the direction of $\hat{\sigma}_+$ is

$$\frac{\partial\psi_+^M}{\partial\sigma_+} = -\frac{\mathrm{i}}{2}\exp(-\mathrm{i}\omega t)\, a_0\,\omega\sin(\mu-\phi_0)\int_0^\infty J_1(K)\exp\left(\mathrm{i}K\frac{\sigma_+}{a}\right)\mathrm{d}K, \tag{4.4}$$

where the superscript M denotes Makarov *et al.* (1990). Also, in (4.4), t is measured from $t_0 = -\pi/\omega$ to change the sign preceding its right-hand side. Now

$$\begin{aligned} a_0\,\omega\sin(\mu-\phi_0) &= a_0\,\omega(\sin\mu\cos\phi_0 - \cos\mu\sin\phi_0) \\ &= U\sin\mu - V\cos\mu. \end{aligned} \tag{4.5}$$

This yields

$$\frac{\partial\psi_+^M}{\partial\sigma_+} = -\frac{\mathrm{i}}{2}\exp(-\mathrm{i}\omega t)\,(U\sin\mu - V\cos\mu)\int_0^\infty J_1(K)\exp\left(\mathrm{i}K\frac{\sigma_+}{a}\right)\mathrm{d}K. \tag{4.6}$$

The corresponding result in our solution with the time factor $\exp(-\mathrm{i}\omega t)$ added is, by (3.42),

$$\frac{\partial\psi_+}{\partial\sigma_+} = -\frac{\mathrm{i}}{2}\exp(-\mathrm{i}\omega t)\,(U\sin\mu - V\cos\mu + \mathrm{i}(U\cos\mu + V\sin\mu))\int_0^\infty J_1(K)\exp\left(\mathrm{i}K\frac{\sigma_+}{a}\right)\mathrm{d}K. \tag{4.7}$$

Equations (4.6) and (4.7) show that there is a significant difference between our solution and that of Makarov *et al.* (1990). Further, for a point $x = a\cos\phi$, $y = b\sin\phi$ on the surface of the cylinder, (4.6) gives

$$\psi_+^M = \tfrac{1}{2}a(V\cos\phi - U\sin\phi). \tag{4.8}$$

Comparison of this last equation and (4.1) shows that ψ_+^M does not take the appropriate values on the surface of the cylinder.

The discrepancies arise because putting $\nu = 0$ in the solution of Makarov *et al.* (1990) does not give the inviscid solution discussed herein. Further comparisons of their work and ours will be made in Part 2.

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Appendix. Verification of boundary condition

The equation of the ellipse given by (2.1) can be written

$$x_0 = a\cos\phi_1, \quad y_0 = b\sin\phi_1, \quad 0 < \phi_1 < 2\pi. \tag{A 1}$$

Equations (A 1), (2.2), (2.3) and figure 3 shows that, if $0 < \theta < \mu$, then, on the ellipse,

$$\sigma_+ = a\cos\phi_1\sin(\mu-\theta) - b\sin\phi_1\cos(\mu-\theta),$$
$$\sigma_- = a\cos\phi_1\sin(\mu+\theta) + b\sin\phi_1\cos(\mu+\theta),$$
$$(\sigma_+^2/c_+^2 - 1)^{1/2} = (\mathrm{i}/c_+)(a\sin(\mu-\theta)\sin\phi_1 + b\cos(\mu-\theta)\cos\phi_1),$$
$$(\sigma_-^2/c_-^2 - 1)^{1/2} = -(i/c_-)(a\sin(\mu+\theta)\sin\phi_1 - b\cos(\mu+\theta)\cos\phi_1).$$

These equations, with (2.2), (2.7), (3.25) and (A 1), then show that the boundary condition is satisfied provided, for $0 < \phi_1 < 2\pi$,

$$a\cos\phi_1(V\cos\theta - U\sin\theta) - b\sin\phi_1(V\sin\theta + U\cos\theta) \equiv \alpha_+ C_+ + \alpha_- C_-, \tag{A 2}$$

where

$$C_+ = (a\cos\phi_1\sin(\mu-\theta) - b\sin\phi_1\cos(\mu-\theta) - \mathrm{i}(a\sin\phi_1\sin(\mu-\theta) + b\cos\phi_1\cos(\mu-\theta))),$$
$$C_- = (a\cos\phi_1\sin(\mu+\theta) + b\sin\phi_1\cos(\mu+\theta) + \mathrm{i}(a\sin\phi_1\sin(\mu+\theta) - b\cos\phi_1\cos(\mu+\theta))).$$

Equating the coefficients of $\sin\phi_1$ and $\cos\phi_1$ in (A 2) leads to equations whose solution for α_+ and α_- is given by (3.26) and (3.27).

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